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MINIMUM ALTITUDE VARIATION ORBITS. ANALYSIS OF CHARACTERISTICS AND STABILITY

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У статті розглядаються закономірності руху супутника на майже кругових орбітах під дією другої зональної гармоніки геопотенціалу. Метою досліджень є визначення параметрів орбіт з мінімальною зміною радіуса і вивчення властивостей цих орбіт. Показано, що задача визначення параметрів орбіт з мінімальною зміною радіуса представляє теоретичний і практичний інтерес. Ці орбіти є найбільш близькими до кеплерових кругових орбіт. Практичний інтерес до таких орбіт визначається можливістю їх використання для наукових досліджень і систем спостереження Землі. На основі аналізу літератури зроблено висновок, що до теперішнього часу ще не повністю побудоване рішення даної задачі: алгоритм визначення параметрів орбіт не цілком обґрунтований і надмірно ускладнений; відсутній аналітичний аналіз стійкості орбіт мінімальної зміни радіуса. Показана ефективність застосування розробленої раніше теорії опису руху супутників на майже кругових орбітах для визначення параметрів орбіт з мінімальною зміною радіуса. З цією метою доопрацьовано рішення першого наближення руху супутників на майже кругових орбітах при впливі другої зональної гармоніки геопотенціалу. Ці рішення дозволяють легко визначати параметри орбіт мінімальної зміни радіуса. Побудовано усереднені рівняння другого наближення впливу другої зональної гармоніки на рух супутника і на їх основі доведено стійкість орбіт з мінімальною зміною радіуса. Показано, що друге наближення за малими параметрами повністю описує основні закономірності довгоперіодичного руху супутника при впливі другої зональної гармоніки геопотенціалу. За допомогою числових досліджень показана нестійкість орбіт з мінімальною зміною радіуса при врахуванні впливу більш високого порядку гармонік геопотенціалу. Аналіз області можливого застосування орбіт мінімальної зміни радіуса показав, що практичне значення такі орбіти можуть мати для дуже низьких і наднизьких орбіт, де керуючий вплив на рух супутника здійснюється не рідше, ніж раз на дві доби.

Ключові слова: *орбіти мінімальної зміни висоти, друга зональна гармоніка геопотенціалу, стійкість орбіти, закономірності руху.*

The article discusses the regularities of satellite motion in almost circular orbits under the influence of the second zonal harmonic of the geopotential. The aim of the research is to determine the parameters of orbits with a minimum change in radius and to study the properties of these orbits. It is shown that the problem of determining the parameters of orbits with a minimum change in radius is of theoretical and practical interest. These orbits are the closest to Keplerian circular orbits. The practical interest in such orbits is determined by the possibility of using them for scientific research and Earth observation systems. Based on the analysis of the literature, it was concluded that the solution of the problem under consideration is not complete by now: the algorithm for determining the parameters of the orbits are not well founded and unnecessarily complicated; there is no analytical analysis of the stability of the orbits of the minimum change in radius. The efficiency of application of the previously developed theory of describing the motion of satellites in almost circular orbits for determining the parameters of orbits with a minimum change in radius is shown. For this purpose, the solutions of the first approximation of the motion of satellites in almost circular orbits under the influence of the second zonal harmonic of the geopotential have been improved. These solutions make it easy to determine the parameters of the orbits of the minimum change in radius. The averaged equations of the second approximation of the influence of the second zonal harmonic on the satellite motion are constructed and, on their basis, the stability of the orbits with a minimum change in radius is proved. It is shown that the second approximation in small parameters completely describes the main regularities of the long-period satellite motion under the influence of the second zonal harmonic of the geopotential. With the help of numerical studies, the instability of orbits with a minimum change in radius is shown with allowance for the effect of higher order harmonics of the geopotential. Analysis of the area of possible application of orbits with a minimum change in radius showed that such orbits can be of practical importance for very low and ultra low orbits, where the control action on the satellite movement is carried out at least once every two days.

Keywords: *minimum altitude variation orbits, second zonal harmonic of the geopotential, stability of the orbits, regularities of motion.*

Introduction. The choice of a satellite orbit corresponding to its mission is an important task, the successful solution of which can significantly increase the efficiency of satellite use. Orbits with a minimum radius variation, or, following [1, 2, 3], orbits with a minimum altitude variation (OMAV) are of theoretical and practical interest. These orbits are the closest to Keplerian circular orbits. Practical interest in OMAV is determined by the possibility of their use for scientific research and Earth observation systems [1, 3, 4].

The main disturbing effect on the satellite's motion in low near-earth orbits is the difference between the Earth's gravitational field and the central one. The potential of the off-center gravitational field of the Earth can be described using the expansion in spherical functions

$$U = \frac{\mu}{R}\left[1 + \sum_{n=2}^{\infty}\left(\frac{R_E}{R}\right)^n\left(C_{n0}P_n(\sin\delta) + \sum_{m=1}^{n}P_{nm}(\sin\delta)(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\right)\right] \quad (1)$$

where μ – gravitational constant of the Earth; R – the distance from the center of the Earth to the considered point in space with geocentric latitude δ and longitude λ in the coordinate system associated with the Earth; R_E – the average equatorial radius of the Earth; C_{n0}, C_{nm}, S_{nm} – dimensionless coefficients depending on the distribution of the Earth's masses; $P_n(\sin\delta)$ – Legendre polynomials of order n ; $P_{nm}(\sin\delta)$ – associated Legendre functions of order n and index m .

The members of expression (1) containing $P_n(\sin\delta)$ are called the second, third, etc. zonal harmonics, and the terms containing $P_{nm}(\sin\delta)$ - sectorial (at $n = m$) and tesseral (at $0 < m < n$) harmonics. The geometrical meaning harmonics is detailed in [5].

The coefficient at the second zonal harmonic C_{20} is three orders of magnitude higher than other expansion coefficients (1), and the effects of the second zonal harmonic describing the compression of the Earth are decisive in the difference between the trajectory of the satellite and the Kepler orbit.

In [6, 7], a new form of equations for the perturbed motion of a satellite in almost circular orbits is proposed. In [7], the first approximation of the influence of the second zonal harmonic on the motion of satellites was constructed. It is shown that the maximum value of the amplitude of forced oscillations of the orbital radius under the action of the second zonal harmonic can reach 3.5 km at an orbital altitude of 675 km. In this regard, the question arises of choosing the initial conditions for the satellite motion, which ensures the minimum oscillations of the orbital radius under the influence of the second zonal harmonic.

One of the first publications devoted to OMAV is article [1]. The brevity of the presentation of the material and the use of special research methods make it difficult to use the results obtained in [1]. The publication [4] indicates a very small number of studies on this issue. It seems that an interesting formulation of the problem of searching for orbits that give the minimum change in the satellite altitude at a given latitude range is considered in [4]. However, the research method based on the use of averaged equations and numerical studies of oscillations of the orbital radius on one revolution does not seem to correspond to the research goal. Studies of OMAV are absent in monographs [5, 8, 9]. Quite clear results on the OMAV are contained in the article [3]. The article is distinguished by the practical orientation of the materials and the possibility of reproducing the results. However, the article uses the final formulas of previous studies, some of which are difficult to access (for example [2]), and the physics of the process is not described. At the same time, the determination of the parameters of the OMAV is, as will be shown below, an unnecessarily complicated process. In [3], there is no analytical analysis of the stability of OMAV.

Thus, the problem of determining the parameters of OMAV is of interest for the design of remote sensing satellites, and its solution is currently not complete. The article proposes a simple way to determine the parameters of the OMAV, and an analysis of the stability of such orbits is carried out.

Problem statement. The motion of the center of mass of the Earth's satellite is considered, taking into account its perturbations only by the second zonal harmonic of the Earth's gravitational field.

The perturbing accelerations acting on the satellite have the form [7]

$$F_1 = -\frac{3C_{20}\mu R_E^2}{2R^4}(3\sin^2 u \sin^2 i - 1),$$

$$F_2 = \frac{3C_{20}\mu R_E^2}{2R^4}\sin 2u \sin^2 i,$$

$$F_3 = \frac{3C_{20}\mu R_E^2}{2R^4}\sin u \sin 2i,$$

where F_1, F_2, F_3 are radial, transversal and normal accelerations respectively, i, u are inclination and latitude argument respectively.

Let us write the equations of satellite motion in the form [7]

$$\begin{aligned}
&i' = -\frac{\varepsilon}{2}\frac{1}{z^3 s^{1/2}}\sin 2u \sin 2i, \quad \Omega' = -2\varepsilon\frac{1}{z^3 s^{1/2}}\cos i \sin^2 u,\\
&\Delta u' = \left(\frac{s^{1/2}}{z^2} - 1\right) - \Omega'\cos i, \quad \gamma' = -2\varepsilon\frac{s^{1/2}}{z^3}\sin^2 i \sin 2u,\\
&b_1' = b_2, \qquad b_2' = \frac{\gamma - b_1}{z^3} + \varepsilon\frac{1}{z^4}(3\sin^2 i \sin^2 u - 1),
\end{aligned} \tag{2}$$

where the prime denotes the derivative with respect to $\tilde{u}$ ($\tilde{u}$ is the argument of the latitude of the unperturbed orbit, $\dot{\tilde{u}} = \sqrt{\frac{\mu}{R_0^3}}$, R_0 – the radius of the unperturbed circular reference orbit); $\varepsilon = -\frac{3}{2}C_{20}\frac{R_E^2}{R_0^2}$; Ω – ascending node longitude; dimensionless variables b_1, b_2, γ are related to the current position and speed of the satellite by the relations

$$R = R_0(1 + b_1), \quad \dot{R} = b_2\sqrt{\mu/R_0}, \quad p = R_0(1+\gamma)$$

where p is the focal parameter of the orbit; $z = 1 + b_1$ is dimensionless orbital radius equal to the ratio of the orbital radius to the reference orbit radius; $s = 1 + \gamma$ is dimensionless focal parameter of the orbit, equal to the ratio of the focal parameter of the orbit to the focal parameter of the reference orbit; $\Delta u = u - \tilde{u}$.

It is necessary to determine the existence of orbits with a minimum radius variation and their properties.

Determining the orbits of the minimum radius variation. Since almost circular orbits are considered, then b_1, b_2, γ are small quantities, the order of which does not exceed an order of magnitude ε (for R_0=7000 km, $\varepsilon = 1.35 \cdot 10^{-3}$). We obtain the equations of the first approximation, keeping in equations (2) only the quantities of the first order of smallness [7]

$$\begin{aligned} &i' = -0.5\varepsilon \sin 2\tilde{u} \sin 2i, \qquad \Omega' = -2\varepsilon \cos i \sin^2 \tilde{u}, \\ &\Delta u' = 0.5\gamma - 2b_1 - \Omega' \cos i, \quad \gamma' = -2\varepsilon \sin^2 i \sin 2\tilde{u}, \\ &b_1' = b_2, \qquad b_2' = \gamma - b_1 + \varepsilon(3\sin^2 i \sin^2 \tilde{u} - 1). \end{aligned} \tag{3}$$

Without loss of generality, we assume that at the initial moment of time $u = \tilde{u} = 0$, i.e. the trajectory "starts" at the ascending node of the orbit. Then, with the considered accuracy $\varepsilon^* = \max\{\varepsilon, b_1, b_2, \gamma\}$, one can find the following solutions [7]

$$\Delta i = i - i_0 = \frac{\varepsilon}{4} \sin 2i_0 (\cos 2\tilde{u} - 1),$$

$$\Delta\Omega = \Omega - \Omega_0 = -\frac{\varepsilon}{2} \cos i_0 \left(2\tilde{u} - \sin 2\tilde{u}\right),$$

$$\Delta\gamma = \gamma - \gamma_0 = \varepsilon \sin^2 i_0 (\cos 2\tilde{u} - 1), \tag{4}$$

where the subscript "0" denotes the initial values of the variables.

The equation of change b_1 is written in the form [7]

$$b_1'' + b_1 = -\frac{\varepsilon}{2} \sin^2 i_0 \cos 2\tilde{u} + \gamma_0 + \varepsilon \left(\frac{1}{2} \sin^2 i_0 - 1 \right). \tag{5}$$

Taking that

$$\gamma_0 = \varepsilon \left(1 - \frac{1}{2} \sin^2 i_0 \right), \tag{6}$$

we obtain that equation (5) describes harmonic oscillations b_1 relative to the zero position. Condition (6) is not a restriction on the values of the focal parameter (transverse satellite velocity) due to the absence of restrictions on the radius of the reference orbit R_0.

Then, the equation describing the change will take the form

$$b_1'' + b_1 = -\frac{\varepsilon}{2} \sin^2 i_0 \cos 2\tilde{u}.$$

We write its solution in the form

$$b_1 = b_{10} \cos\tilde{u} + b_{10}' \sin\tilde{u} + \frac{d}{3}\left(\cos 2\tilde{u} - \cos\tilde{u}\right) = A\cos(\tilde{u} - \alpha) + \frac{d}{3}\left(\cos 2\tilde{u} - \cos\tilde{u}\right), \tag{7}$$

where $d = \frac{\varepsilon}{2}\sin^2 i_0$; A, α are amplitude and phase shift of natural oscillations, $d/3$ is amplitude of forced oscillations, b_{10}, b'_{10} are the initial conditions for changing the orbit radius at its ascending node.

In the general case, expression (7) is doubly periodic: oscillations with a doubled orbital frequency are added to oscillations with an orbital frequency. It is easy to understand that the minimum deviation b_1 from zero is achieved when the amplitude of oscillations with the orbital frequency is equal to zero. Consequently, for $b_{10} = \frac{1}{6}\varepsilon\sin^2 i_0$ and $b'_{10} = 0$, the change in the radius of the orbit will be minimal and equal to

$$\Delta R = \frac{1}{6} R_0 \varepsilon \sin^2 i_0 \cos 2\tilde{u}, \qquad \max \Delta R = -\frac{1}{4} C_{20} \sin^2 i_0 R_E \frac{R_E}{R_0}.$$

For an orbit with an altitude of ≈500 km, the amplitude of the radius change will be ≈1.6 km ($C_{20} = -1.0826 \cdot 10^{-3}$, $R_E = 6378.1363$ km).

The change in the radius of the orbit in this case is described by the equality

$$b_1 = \frac{d}{3}\cos 2\tilde{u},$$

and has the property of symmetry about the center of mass of the Earth.

Let us complete the construction of the solution to equations (3). Substituting (7), (4) with condition (6) into the equation of change $\Delta u = u - \tilde{u}$ and integrating, we obtain

$$\begin{aligned} \Delta u = 0.5\varepsilon(3 - 3.5\sin^2 i)\tilde{u} + 0.5\varepsilon\sin 2\tilde{u}\left(\frac{7}{6}\sin^2 i - 1\right) + \\ + \frac{1}{3}\varepsilon\sin^2 i \sin\tilde{u} - 2A\left(\sin(\tilde{u} - \alpha) + \sin\alpha\right). \end{aligned} \tag{8}$$

Let us estimate the period of the satellite's motion between two successive intersections of the ascending node – the nodal (draconian) period. Let for the time when u changes by 2π the change in $\tilde{u}$ is equal to $2\pi - \delta$. Then the change in $\Delta u = \delta$ for the same time. Consequently, δ is small, and to estimate the nodal period with accuracy up to the square of small values from (8) we obtain $\delta = \varepsilon(3 - 3.5\sin^2 i)\pi$.

Consequently, during the nodal period $\tilde{u}$ change by the value $2\pi(1 - 0.5\varepsilon(3 - 3.5\sin^2 i))$. The time of such a change is equal to

$$P_\Omega = \sqrt{\frac{R_0^3}{\mu}}\, 2\pi(1 - 0.5\varepsilon(3 - 3.5\sin^2 i)),$$

where P_Ω is the nodal period.

For comparison with the well-known formulas [5, 8, 9], we note that, up to a square of small values, the semi-major axis of the orbit a is equal to

$$a = \frac{R_0}{1-\gamma_0}.$$

Consider **changing the trajectory depending on** u **.** For this, in the equations of perturbed motion (2), we pass from the independent variable $\tilde{u}$ to the independent variable u . In the general case, such a transition leads to the fact that the right-hand sides of the equations will differ in the presence of an additional multiplier $\left(\frac{s^{1/2}}{z^2} - \mathrm{ctg}\, i \sin u \tilde{F}_3\right)^{-1}$, where $\tilde{F}_3 = \frac{R_0^2}{\mu s^{1/2}} F_3$.

Expanding this multiplier in a series in small quantities keeping only the first order of smallness, we obtain

$$\left(\frac{s^{1/2}}{z^2} - \mathrm{ctg}\, i \sin u \tilde{F}_3\right)^{-1} \approx (1 + 0.5\gamma - 2b_1 - \mathrm{ctg}\, i \sin u \tilde{F}_3)\,.$$

Consequently, the equations of the first approximation (3) will not change when passing to differentiation with respect to u . And, consequently, their decisions will remain in effect when $\tilde{u}$ is replaced by u . Taking into account the physics of the process, replacing $\tilde{u}$ in solutions with u will give a more accurate approximation.

Thus, the change in the radius of the orbit in the first approximation is described by the expression

$$b_1 = b_{10} \cos u + b'_{10} \sin u + \frac{d}{3}(\cos 2u - \cos u) =$$
$$= A \cos(u - \alpha) + \frac{d}{3}(\cos 2u - \cos u)$$

where $b_1 = R/R_0 - 1$; R, R_0 are the radii of the orbit and the reference circular orbit, respectively. The initial conditions of movement for OMAV are $b_{10} = \frac{d}{3}, b'_{10} = 0$, and the change in the radius of the OMAV is described by the equality

$$b_1 = \frac{d}{3} \cos 2u\ .$$

In this case, the initial value of the focal parameter is equal to $p_0 = R_0(1+\gamma_0)$, where γ_0 is determined by the equality (6).

For the convenience of converting into other variables, we present the expressions for the initial radius and speed of the satellite in the ascending node of the OMAV. The radius is

$$R = R_0\left(1 + \frac{\varepsilon}{6} \sin^2 i_0\right),$$

and the velocity is directed strictly along the transversal and is equal to

$$V_\tau = \frac{\sqrt{\mu p_0}}{R} = \sqrt{\frac{\mu}{R_0}}\,\frac{\sqrt{1+0.5\varepsilon(1+\cos^2 i_0)}}{1+\dfrac{\varepsilon}{6}\sin^2 i_0}.$$

Stability analysis of OMAV. The equations of the first approximation do not allow us to draw conclusions about the stability of the OMAV. For theoretical studies of stability, it is necessary to consider the equations of the second approximation. To do this, consider the equation of perturbed motion with an independent variable u

$$\begin{aligned}
&i' = -\frac{\varepsilon}{2}\frac{w}{z^3\sqrt{s}}\sin 2u \sin 2i, \quad \Omega' = -2\varepsilon\frac{w}{z^3\sqrt{s}}\cos i \sin^2 u,\\
&\Delta u' = w\left(\frac{\sqrt{s}}{z^2}-1\right) - \Omega'\cos i, \quad \gamma' = -2\varepsilon w\frac{\sqrt{s}}{z^3}\sin^2 i \sin 2u, \qquad (9)\\
&b_1' = wb_2, \qquad b_2' = w\frac{\gamma - b_1}{z^3} + \varepsilon\frac{w}{z^4}(3\sin^2 i \sin^2 u - 1),
\end{aligned}$$

where $w = \left(\dfrac{\sqrt{s}}{z^2} + 2\varepsilon\dfrac{1}{z^4\sqrt{s}}\cos^2 i \sin^2 u\right)^{-1}$.

It can be seen from equations (9) that the change in the shape of the orbit (parameters b_1, b_2) does not depend on the changes Ω (due to the axial symmetry of the equatorial hump). Therefore, to study the stability of OMAV, it is sufficient to investigate the equations for i, γ, b_1, b_2.

We introduce new variables: $i = i_1 + i_2$, $\gamma = \gamma_1 + \gamma_2$, where subscripts 1 and 2 denote the components of the solution of equations (9) proportional to the first and second degrees of smallness, respectively.

To describe changes in the shape of the orbit in the second approximation, we introduce new variables as follows

$$b_1 = A\cos(u-\alpha) + d/3(\cos 2u - \cos u),$$
$$b_2 = -A\sin(u-\alpha) + d/3(\sin u - 2\sin 2u).$$

Then

$$A' = -\frac{d}{3}\left[2\cos 2u \sin(u-\alpha) - 2\sin(u+\alpha) + \sin\alpha\right] + b_{1r}\cos(u-\alpha) - b_{2r}\sin(u-\alpha),$$

$$A\alpha' = A + \frac{d}{3}\left[2\cos 2u\cos(u-\alpha) + 2\cos(u+\alpha) - \cos\alpha\right] + b_{1r}\sin(u-\alpha) + b_{2r}\cos(u-\alpha),$$

where b_{1r}, b_{2r} are the right-hand sides of equations (9) for b_1, b_2, respectively.

Let us construct the averaged equations for the second approximation. For this, we substitute new variables into equations (9). Keeping the quantities no higher than the second order of smallness and averaging the resulting equations over u, one can obtain

$$\bar{i}_2 = 0, \bar{\gamma}_2 = 0,$$

$$\bar{A}\bar{\alpha}' = 2\bar{A}\varepsilon - 5\bar{A}d + \frac{5}{3}d^2\cos\bar{\alpha} - \frac{2}{3}d\varepsilon\cos\bar{\alpha} = -\bar{A}G + \frac{d}{3}G\cos\bar{\alpha}, \tag{10}$$

$$\bar{A}' = \frac{1}{3}d\sin\bar{\alpha}(5d - 2\varepsilon) = \frac{d}{3}G\sin\bar{\alpha},$$

where the “hat” denotes the average values; $G = 5d - 2\varepsilon$.

The system of equations (10) can be easily solved by introducing new variables $\lambda = \bar{A}\cos\bar{\alpha}$, $h = \bar{A}\sin\bar{\alpha}$. .The change in these variables is described by the equations

$$\lambda' = Gh,$$

$$h' = \frac{d}{3}G - G\lambda.$$

The sought solutions for $G > 0$ and $G < 0$ can be written in the form

$$\bar{A}\sin\bar{\alpha} = -\left(\bar{A}_0\cos\bar{\alpha}_0 - \frac{d}{3}\right)\sin Gu + \bar{A}_0\sin\bar{\alpha}_0\cos Gu,$$

$$\bar{A}\cos\bar{\alpha} = \frac{d}{3} + \left(\bar{A}_0\cos\bar{\alpha}_0 - \frac{d}{3}\right)\cos Gu + \bar{A}_0\sin\bar{\alpha}_0\sin Gu, \tag{11}$$

where $\bar{A}_0, \bar{\alpha}_0$ are the initial conditions of the averaged equations.

The results of numerical studies show that when G is of order ε, the averaged equations describe well long-period changes in the shape of the orbit. The case when G is of order ε^2 requires additional research. Figure 1 shows the changes in the amplitudes $A, \bar{A}$ and phase shifts $\alpha, \bar{\alpha}$ (apogee arguments) of the complete (2) and averaged equations at 2000 revolutions of the satellite. The solutions of the complete equations shown in the figure by the solid line were carried out numerically. The solutions of the averaged equations shown in the figure by circles are given by formulas (11). The initial conditions of motion were taken as follows: $R_0 = R_{sr} + 500$ km, $R_{sr} = 6371$ km, $i_0 = 98.1°$, $A_0 = d/3$, $\alpha_0 = -10°$, $\gamma_0 = \varepsilon(1 - 0.5\sin^2 i_0)$.

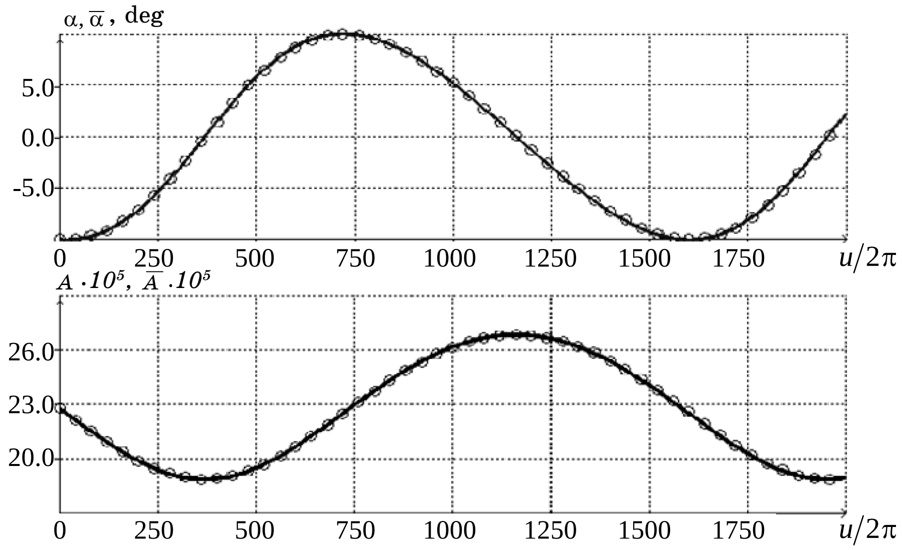

Fig. 1

Calculations show that the second approximation in small parameters completely describes the main regularities of the long-period satellite motion under the influence of the second zonal harmonic of the geopotential except for the cases when $\sin^2 i_0 \approx 0.8$. The averaged equations make it possible to solve the problem of the stability of the orbits with a minimum change in altitude. We rewrite solutions (11) in the form

$$\begin{aligned} \bar{A}\sin\bar{\alpha} &= -B\sin(Gu-\tau), \\ \bar{A}\cos\bar{\alpha} &= \frac{d}{3} + B\cos(Gu-\tau), \end{aligned} \tag{12}$$

where $B^2 = \left(\bar{A}_0 - \frac{d}{3}\right)^2 + 2\bar{A}_0\frac{d}{3}(1-\cos\bar{\alpha}_0)$ is amplitude of long-period oscillations, τ is phase shift of long-period oscillations, $\cos\tau = \frac{\bar{A}_0\cos\bar{\alpha}_0 - \frac{d}{3}}{B}$, $\sin\tau = \frac{\bar{A}_0\sin\bar{\alpha}_0}{B}$.

It is easily seen from (12) that there is only one equilibrium position $\bar{A} = \frac{d}{3}, \bar{\alpha} = 0$. This equilibrium position is stable and, with small deviations, the variables λ, h fluctuate relative to it with an amplitude B. When $B \le \frac{d}{3}$, the apogee fluctuates $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$. When $B > \frac{d}{3}$, the apogee of the orbit rotates.

Scope of possible application of OMAV. OMAV give an estimate of the minimum possible change in satellite heights. In fig. 2, for the initial conditions, $R_0 = R_{sr} + 507$ km, $R_{sr} = 6371$ km, $i_0 = 97.4°$, $A_0 = d/3$, $\alpha_0 = 0$, $\gamma_0 = \varepsilon(1 - 0.5\sin^2 i_0)$, the change in the orbital height relative to the mean radius of the Earth on the first two turns is shown

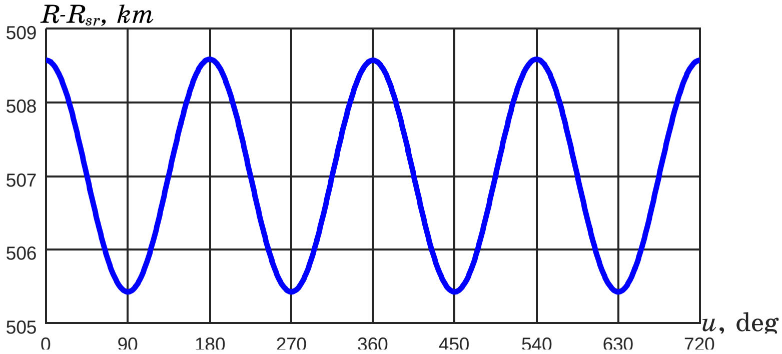

Fig. 2 – Change in orbital height relative to the mean radius of the Earth on the first two turns

For OMAV, the orbital radius changes by almost 3.16 km, or 1.6 km relative to the average. In this case, the maximum values are reached above the equator, and the minimum - above the poles. However due to the flattening of the Earth, the height above the common earth's ellipsoid changes by almost 18.23 km, or 9.1 km relative to the average.

Despite the stability of the OMAV under the action of the second zonal harmonic of the geopotential, the OMAV is not stable under the action of other disturbing forces. Calculations show that the main influence on the destruction of OMAV is exerted by the effects of other zonal harmonics of the geopotential. Figure 3 shows the change in the orbit height relative to the mean equatorial radius, taking into account the 30x30 harmonics of the geopotential at 30 turns.

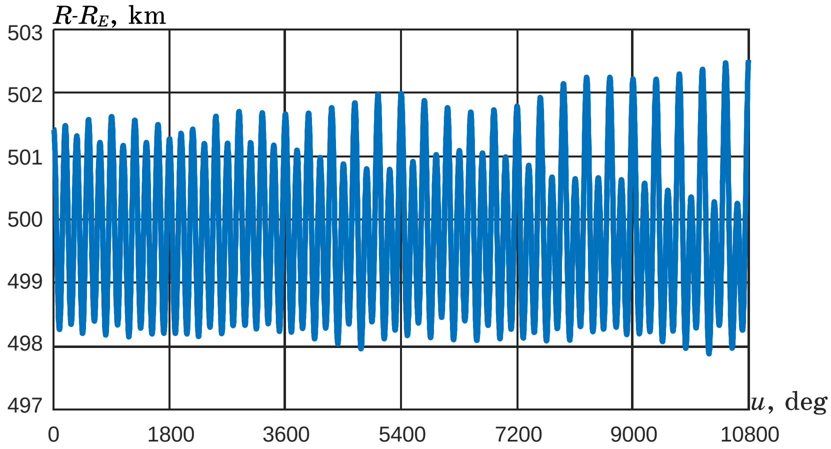

Fig. 3 – Change in orbit height relative to the mean equatorial radius, taking into account 30x30 harmonics of the geopotential at 30 turns

If at the first orbit changes in the shape of the orbit is insignificant, see Fig. 4 (Change in the orbit height relative to the mean equatorial radius, taking into account the 30x30 harmonics of the geopotential on the first two orbits), then by the 30th orbit it becomes significant, see Fig. 5 (Change in the orbit height relative to the mean equatorial radius, taking into account the 30x30 harmonics of the geopotential at the 29th and 30th orbits). Changes in the shape of the orbit are associated with changes in eccentricity and perigee argument.

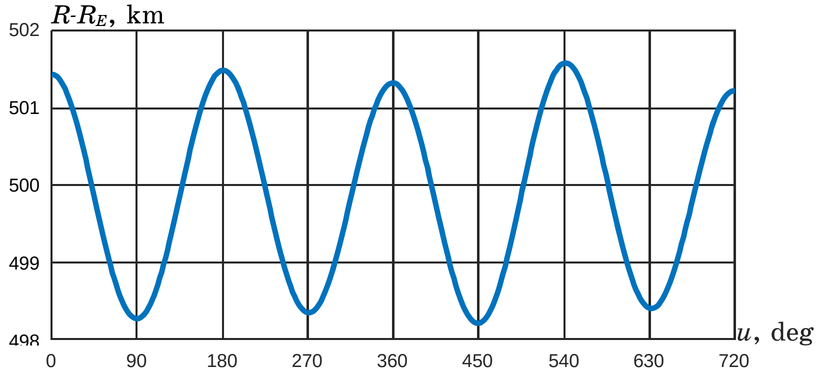

Fig. 4 – Change in the orbit height relative to the mean equatorial radius, taking into account the 30x30 harmonics of the geopotential on the first two orbits

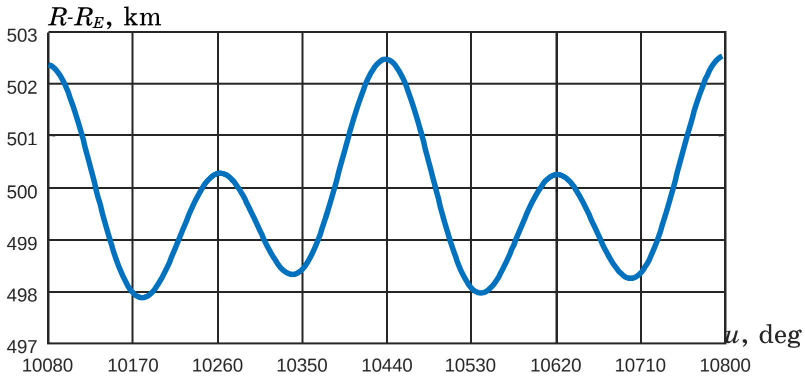

Fig. 5 – Change in the orbit height relative to the mean equatorial radius, taking into account the 30x30 harmonics of the geopotential at the 29th and 30th orbits

Thus, the maintenance of the OMAV is associated with rather frequent (every two days) control actions correcting the orbit. For ultra-low and very low orbits, where compensation of aerodynamic braking requires frequent activation of the correcting thrusters, the OMAV can be selected as the reference orbits.

Conclusions.

1. The constructed orbits of the minimum altitude variation (OMAV) are of theoretical interest, since they determine the minimum possible changes in the heights of the orbits. For orbits with an altitude of about 500 km, these changes are: about 1.6 km change relative to mean radius; about 9.1 km relative to the mean orbital altitude above the Earth's common ellipsoid.
2. The effectiveness of the application of the previously developed theory of describing the motion of satellites in almost circular orbits for determining the parameters of OMAV is shown.
3. The averaged equations of the second approximation of the influence of the second zonal harmonic on the motion of the satellite are constructed and the stability of the OMAV is proved on their basis. It is shown that the second approximation in small parameters, with the exception of the cases when $\sin^2 i_0 \approx 0.8$, fully describes the main regularities of the long-period satellite motion under the influence of the second zonal harmonic of the geopotential.
4. Using numerical studies, the instability of the OMAV is shown when taking into account the influence of higher-order geopotential harmonics. Preliminary calculations show that during the two days of flight, the characteristics of the OMAV are already significantly distorted.
5. OMAV can be of practical importance for very low and ultra-low orbits, where the control action on the satellite movement is carried out at least once every two days.

1. *Rider L.* Class of Minimum Altitude Variation an Oblate Earth. ARS Journal. 1961. Vol. 31. No 11. P. 1580–1582.
2. *Пшеничников В. В., Лысов Н. С., Бабяк А. П.* Результаты исследования характеристик орбиты минимального изменения высоты. ЦИВТИ,1978. Указатель № 3 (137).
3. *Артюшенко В. М., Виноградов Д. Ю.* Анализ свойств орбит минимального изменения высоты. Информационно-технологический вестник. 2017. № 4 (14). С. 3–15. https://doi.org/10.21499/2409-1650-2017-4-3-15
4. *Culp R. D., Murrow R. C.* Minimum Altitude Variation Arcs. SAE Transactions. 1986. Vol. 95. P. 661–670. https://doi.org/10.4271/861665

5. *Vallado D. A.* Fundamentals of astrodynamics and applications. Fourth Edition. Space Technology Library. 2013. 1106 p. ISBN-13: 978-1881883180.
6 *Pirozhenko A., Maslova A., Khramov D., Volosheniuk O., Mischenko A.* Development of a new form of equations of disturbed motion of a satellite in nearly circular orbits. Eastern-European Journal of Enterprise Technologies. 2020. Vol. 4. N 5 (106). P. 70–77. https://doi.org: 10.15587/1729-4061.2020.207671
7. *Пироженко А. В., Маслова А. И., Васильев В. В.* О влиянии второй зональной гармоники на движение спутника по почти круговым орбитам. Космічна наука і технологія. 2019. Т. 25, № 2. С. 3–14. https://doi.org/10.15407/knit2019.02.003
8. *Эльясберг П. Е.* Введение в теорию полета искусственных спутников Земли. Ленанд, 2015. 544 с. ISBN 978-5-9710-1747-9.
9. *Beutler G.* Methods of celestial mechanics Vol. II: Application to Planetary System, Geodynamics and Satellite Geodesy. Springer-Verlag Berlin Heidelberg, 2005. 468 p. ISBN 978-3-540-26512-2. https://doi.org/10.1007/b137725

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